Longman Scientific & Technical,
Longman Group UK Limited,
Longman House, Burnt Mill, Harlow,
Essex CM20 2JE, England
and Associated Companies throughout the world.

Copublished in the United States with
John Wiley & Sons, Inc., 605 Third Avenue, New York, NY 10158

First published 1991

AMS Subject Classification: (Main) 35P, 45A, 45M
(Subsidiary) 35L, 81F, 45L

ISSN 0269-3674

British Library Cataloguing in Publication Data
Integral equations and inverse problems.
1. Integral equations
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515.45

ISBN 0-582-07766-4

Library of Congress Cataloging-in-Publication Data
Integral equations and inverse problems / Vesselin Petkov and Raitcho Lazarov, editors.
p. cm.—(Pitman research notes in mathematics series ; 235)
Papers presented at the International Conference on Integral Equations and Inverse Problems, held in Varna, Bulgaria, 1989.
ISBN 0-470-21674-3
1. Integral equations--Congresses. 2. Inverse problems
I. Petkov, Veselin. II. Lazarov, Raitcho.
III. International Conference on Integral Equations and Inverse Problems (1989: Varna, Bulgaria)
IV. Series.
QA431.I478 1990
515'.45--dc20

90-5575
CIP

Printed and bound in Great Britain
by Biddles Ltd, Guildford and King's Lynn

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Integral Equations and Inverse Problems

Longman Scientific & Technical

Copublished in the United States with
John Wiley & Sons, Inc., New York

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One can hear the dimension of a connected fractal in R^2

Let Ω be a bounded domain in R^2 with boundary Γ; it is assumed that Γ has zero measure. We consider an eigenvalue problem

$$-\Delta u = \lambda u, \tag{1}$$

$$u|_{\Gamma} = 0, \tag{2}$$

where $\Delta = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2$ is the Laplacian acting on Ω. Problem (1), (2) is posed in variational form on functions from $H_0^1(\Omega)$.

Problem (1), (2) has a discrete spectrum $\lambda_1 \le \lambda_2 \le \ldots \le \lambda_k \le \ldots$. By $N(\lambda)$ we shall denote the eigenvalue distribution function, i.e. the number of eigenvalues λ_k below a given λ.

It is known [1] that the following asymptotics holds

$$N(\lambda) = \frac{S}{4\pi}\lambda + o(\lambda), \quad \lambda \to +\infty, \tag{3}$$

where $S = |\Omega|$ is the area (Lebesgue measure) of Ω. Thus the area of Ω is uniquely recovered from the spectrum. The aim of this work is to show that a certain real number $1 \le d^{(i)} \le 2$ having the sense of dimension and characterizing the degree of 'hairiness' of the boundary from the interior is also uniquely reconstructed from the spectum. Such 'hairy' objects are called fractals [2].

Let us give some definitions.

By Γ_ε we shall denote the ε-neighbourhood of set Γ. The number d defined as the infimum of all positive δ such that

$$\limsup_{\varepsilon \to +0} \varepsilon^{\delta - 2} |\Gamma_\varepsilon| < +\infty \tag{4}$$

is called [3], [4] the Bouligand-Minkowski dimension of Γ.

By $\Gamma_\varepsilon^{(i)}, \Gamma_\varepsilon^{(e)}$ we shall denote subsets of Γ_ε determined by relations $x \in \Omega$, $x \bar{\in} \Omega$ respectively. Substituting in formula (4) Γ_ε by $\Gamma_\varepsilon^{(i)}, \Gamma_\varepsilon^{(e)}$ we define by analogy with d numbers $d^{(i)}, d^{(e)}$. We will call these numbers respectively the interior and exterior Bouligand-Minkowski dimension of the boundary.

Finally by h we shall denote the Hausdorff dimension of the boundary defined [2] as the infimum of positive η such that

$$\lim_{\varepsilon \to +0} \Big(\inf \sum_{j \in J} r_j^{\eta}\Big) = 0. \tag{5}$$

The infimum in (5) is taken over all coverings of set Γ by balls of radii $r_j < \varepsilon$.

It can be shown (see also [3], [4]) that

$$d, d^{(i)}, d^{(e)}, h \in [1,2], \quad d = \max(d^{(i)}, d^{(e)}), \quad h \le d. \tag{6}$$

Let us introduce the function

$$Z(t) = \sum_{k=1}^{+\infty} \exp(-\lambda_k t)$$

$t > 0$. It is known [5] that $Z(t) < S/4\pi t$.

We will call a compact set in R^2 connected if it cannot be divided into two non-empty subsets with a smooth compact curve separating them.

The principal result of this work is contained in the following

Theorem. If Γ has only a finite number of connected components then

$$-2 \liminf_{t \to +0} \frac{\ln(S/4\pi t - Z(t))}{\ln t} = d^{(i)}. \tag{7}$$

Let us give an outline of the theorem's proof. Let $u(x, y, t)$ denote the solution of the heat equation $+\Delta u = \partial u/\partial t$ with boundary condition (2) and initial condition $u|_{t=0} = \delta(x-y)$, $y \in \Omega$. Let $u_0(x, y, t) = (4\pi t)^{-1} \exp(-|x-y|^2/4t)$ denote the fundamental solution of the heat equation. Of course

$$Z(t) = \int_\Omega u(x, x, t)\,dx, \tag{8}$$

$$0 \le u(x, y, t) < u_0(x, y, t), \quad x, y \in \Omega, \quad t > 0. \tag{9}$$

Let $\rho(x, \Gamma)$ denote the distance from x to Γ. It can be shown that for sufficiently small t

$$u_0(x, x, t) - u(x, x, t) \le c_{\kappa\mu} t^{\mu}, \quad \rho(x, \Gamma) \ge t^{\kappa} \quad (\forall \kappa < 1/2, \forall \mu > 0), \tag{10}$$

$$u_0(x, x, t) - u(x, x, t) > C t^{-1}, \quad \rho(x, \Gamma') \le t^{1/2}. \tag{11}$$

In (11) C is a universal positive constant indepndent of Ω and $\Gamma' \neq \emptyset$ is a connected component of Γ with dimension $d^{(i)}$. Substituting (9)-(11) into (8) and using the definition of dimension $d^{(i)}$ we obtain (7).

The most delicate point in the arguments outlined above is obtaining estimate (11). It is interesting that in the case $\Omega \subset R^n$, $n > 3$, an analogous estimate $u_0 - u \geq C t^{-n/2}$ does not hold and in place of (7) we have

$$- 2 \liminf_{t \to +0} \frac{\ln(|\Omega|/(4\pi t)^{n/2} - Z(t))}{\ln t} \leq d^{(i)} \tag{12}$$

with effective examples of strict inequality. We must note that a somewhat weaker version of inequality (12) with d instead of $d^{(i)}$ (see also (6)) in the right-hand side was obtained in [4].

In conclusion let us discuss Berry's conjecture [6] on the existence of a two-term asymptotic expansion

$$N(\lambda) = a\lambda^{n/2} + b\lambda^{\nu/2} + o(\lambda^{\nu/2}), \quad \lambda \to +\infty, \tag{13}$$

$a = (2\pi)^{-n} B_n |\Omega|$, B_n is the volume of the unit ball in R^n, ν is the 'dimension' of Γ, b is a certain constant characterizing the ν-dimensional 'volume' of Γ.

The first problem arising in connection with (13) is how to define dimension ν? Berry proposed $\nu = h$ but a counterexample was constructed in [3]. It was proposed [3], [4] to take $\nu = d$. But it follows from (7), (12) that for the general case $\nu = d$ also is not good because one can easily construct examples of domains Ω with $d^{(i)} < d$ (in this situation a contradiction can be avoided only if $b = 0$ which is rather an unnatural assumption).

Another remark on Berry's conjecture is that the form const $\cdot\ \lambda^{\nu/2}$ of the second asymptotic term is not obvious. If one considers a domain Ω with a self-similar [2] boundary Γ one can expect the two-term $N(\lambda)$ asymptotics to have the form

$$N(\lambda) = a\lambda^{n/2} + f(\ln \lambda)\lambda^{\nu/2} + o(\lambda^{\nu/2}),$$

where $f(\cdot)$ is a periodic function with period $2 \ln s$, $s > 1$ is the linear similarity coefficient, $\nu = d = d^{(i)} = d^{(e)}$.

References

[1] J. Fleckinger and G. Métivier, Théorie spectrale des opérateurs uniformément elliptiques sur quelques ouverts irréguliers, C.R. Acad. Sci. Paris, Ser. A, v. 276, 1973, p. 913-916.

[2] B.B. Mandelbrot, The fractal geometry of nature, Freeman, San Francisco, 1982.

[3] J. Brossard and R. Carmona, Can one hear the dimension of a fractal? Comm. Math. Phys., 1986, v. 104, p. 103-122.

[4] M.L. Lapidus and J. Fleckinger, Tambour fractal: vers une resolution de la conjecture de Weyl-Berry pour valeurs propres du laplacien, C.R. Acad. Sci. Paris, Ser. I Math., 1988, v. 306, p. 171-175.

[5] R. Seeley, A sharp asymptotic remainder estimate for the eigenvalues of the Laplacian in a domain of R^3, Adv. Math., 1978, v. 29, p. 244-269.

[6] M.V. Berry, Some geometric aspects of wave motion: wavefront dislocations, diffraction catastrophes, diffractals, in Geometry of the Laplace Operator; Proc. Symp. Pure Math., v. 36, Amer. Math. Soc., 1980, p. 13-38.

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